APPENDIX D

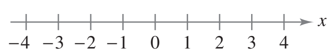
Precalculus Review

APPENDIX D.1 Real Numbers and the Real Number Line

Real Numbers and the Real Number Line • Order and Inequalities • Absolute Value and Distance

Real Numbers and the Real Number Line

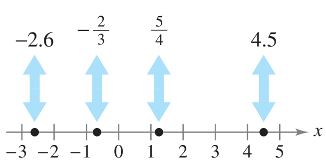
Real numbers can be represented by a coordinate system called the **real number line** or x-axis (see Figure D.1). The real number corresponding to a point on the real number line is the **coordinate** of the point. As Figure D.1 shows, it is customary to identify those points whose coordinates are integers.



The real number line
Figure D.1

The point on the real number line corresponding to zero is the **origin** and is denoted by 0. The **positive direction** (to the right) is denoted by an arrowhead and is the direction of increasing values of x. Numbers to the right of the origin are **positive.** Numbers to the left of the origin are **negative.** The term **nonnegative** describes a number that is either positive or zero. The term **nonpositive** describes a number that is either negative or zero.

Each point on the real number line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real number line. This type of relationship is called a **one-to-one-correspondence.**



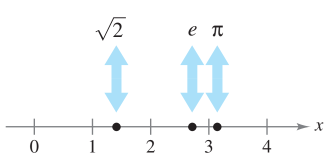
Rational numbers
Figure D.2

Each of the four points in Figure D.2 corresponds to a **rational number**—one that can be written as the ratio of two integers. $\left(\text{Note that } 4.5 = \frac{9}{2} \text{ and } -2.6 = -\frac{13}{5}.\right)$ Rational numbers can be represented either by *terminating decimals* such as $\frac{2}{5} = 0.4$, or by *repeating decimals* such as $\frac{1}{3} = 0.333 \ldots = 0.\overline{3}$.

Real numbers that are not rational are **irrational.** Irrational numbers cannot be represented as terminating or repeating decimals. In computations, irrational numbers are represented by decimal approximations. Here are three familiar examples.

$$\sqrt{2} \approx 1.414213562$$
$$\pi \approx 3.141592654$$
$$e \approx 2.718281828$$

(See Figure D.3.)



Irrational numbers
Figure D.3

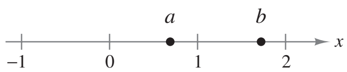
Order and Inequalities

One important property of real numbers is that they are **ordered.** If a and b are real numbers, a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

This relationship can also be described by saying that b is **greater than** a and writing $b > a$. When three real numbers a, b, and c are ordered such that $a < b$ and $b < c$, you say that b is **between** a and c and $a < b < c$.

Geometrically, $a < b$ if and only if a lies to the *left* of b on the real number line (see Figure D.4). For example, $1 < 2$ because 1 lies to the left of 2 on the real number line.



$a < b$ if and only if a lies to the left of b.
Figure D.4

The following properties are used in working with inequalities. Similar properties are obtained if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$. (The symbols $\leq$ and $\geq$ mean **less than or equal to** and **greater than or equal to,** respectively.)

Properties of Inequalities

Let a, b, c, d, and k be real numbers.

1. If $a < b$ and $b < c$, then $a < c$. — Transitive Property
2. If $a < b$ and $c < d$, then $a + c < b + d$. — Add inequalities.
3. If $a < b$, then $a + k < b + k$. — Add a constant.
4. If $a < b$ and $k > 0$, then $ak < bk$. — Multiply by a positive constant.
5. If $a < b$ and $k < 0$, then $ak > bk$. — Multiply by a negative constant.

NOTE Note that you *reverse the inequality* when you multiply the inequality by a negative number. For example, if $x < 3$, then $-4x > -12$. This also applies to division by a negative number. So, if $-2x > 4$, then $x < -2$.

A **set** is a collection of elements. Two common sets are the set of real numbers and the set of points on the real number line. Many problems in calculus involve **subsets** of one of these two sets. In such cases, it is convenient to use **set notation** of the form $\{x\text{: condition on } x\}$, which is read as follows.

The set of all x such that a certain condition is true.

$$\{ \quad x \quad : \quad \text{condition on } x\}$$

For example, you can describe the set of positive real numbers as

$\{x\text{: } x > 0\}.$ — Set of positive real numbers

Similarly, you can describe the set of nonnegative real numbers as

$\{x\text{: } x \geq 0\}.$ — Set of nonnegative real numbers

The **union** of two sets A and B, denoted by $A \cup B$, is the set of elements that are members of A *or* B *or both*. The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements that are members of A *and* B. Two sets are **disjoint** if they have no elements in common.

The most commonly used subsets are **intervals** on the real number line. For example, the **open** interval

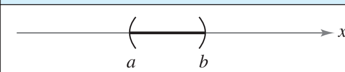
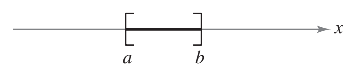
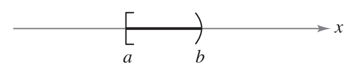
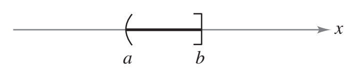
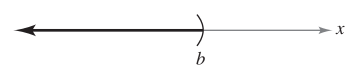
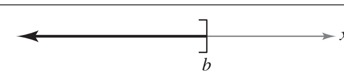
$$(a, b) = \{x: a < x < b\}$$ Open interval

is the set of all real numbers greater than a and less than b, where a and b are the **endpoints** of the interval. Note that the endpoints are not included in an open interval. Intervals that include their endpoints are **closed** and are denoted by

$$[a, b] = \{x: a \leq x \leq b\}.$$ Closed interval

The nine basic types of intervals on the real number line are shown in the table below. The first four are **bounded intervals** and the remaining five are **unbounded intervals.** Unbounded intervals are also classified as open or closed. The intervals $(-\infty, b)$ and (a, ∞) are open, the intervals $(-\infty, b]$ and $[a, \infty)$ are closed, and the interval $(-\infty, \infty)$ is considered to be both open *and* closed.

Intervals on the Real Number Line

	Interval Notation	Set Notation	Graph
Bounded open interval	(a, b)	$\{x: a < x < b\}$	
Bounded closed interval	$[a, b]$	$\{x: a \leq x \leq b\}$	
Bounded intervals (neither open nor closed)	$[a, b)$	$\{x: a \leq x < b\}$	
	$(a, b]$	$\{x: a < x \leq b\}$	
Unbounded open intervals	$(-\infty, b)$	$\{x: x < b\}$	
	(a, ∞)	$\{x: x > a\}$	
Unbounded closed intervals	$(-\infty, b]$	$\{x: x \leq b\}$	
	$[a, \infty)$	$\{x: x \geq a\}$	
Entire real line	$(-\infty, \infty)$	$\{x: x$ is a real number$\}$	

NOTE The symbols ∞ and $-\infty$ refer to positive and negative infinity, respectively. These symbols do not denote real numbers. They simply enable you to describe unbounded conditions more concisely. For instance, the interval $[a, \infty)$ is unbounded to the right because it includes *all* real numbers that are greater than or equal to a.

EXAMPLE 1 Liquid and Gaseous States of Water

Describe the intervals on the real number line that correspond to the temperature x (in degrees Celsius) for water in

a. a liquid state. **b.** a gaseous state.

Solution

a. Water is in a liquid state at temperatures greater than 0°C and less than 100°C, as shown in Figure D.5(a).

$$(0, 100) = \{x: 0 < x < 100\}$$

b. Water is in a gaseous state (steam) at temperatures greater than or equal to 100°C, as shown in Figure D.5(b).

$$[100, \infty) = \{x: x \geq 100\}$$

(a) Temperature range of water (in degrees Celsius)

(b) Temperature range of steam (in degrees Celsius)

Figure D.5

A real number a is a **solution** of an inequality if the inequality is **satisfied** (is true) when a is substituted for x. The set of all solutions is the **solution set** of the inequality.

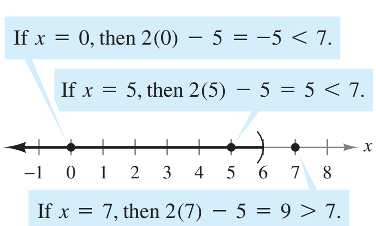
EXAMPLE 2 Solving an Inequality

Solve $2x - 5 < 7$.

Solution

$2x - 5 < 7$	Write original inequality.
$2x - 5 + 5 < 7 + 5$	Add 5 to each side.
$2x < 12$	Simplify.
$\dfrac{2x}{2} < \dfrac{12}{2}$	Divide each side by 2.
$x < 6$	Simplify.

The solution set is $(-\infty, 6)$.



Checking solutions of $2x - 5 < 7$

Figure D.6

NOTE In Example 2, all five inequalities listed as steps in the solution are called **equivalent** because they have the same solution set.

Once you have solved an inequality, check some x-values in your solution set to verify that they satisfy the original inequality. You should also check some values outside your solution set to verify that they *do not* satisfy the inequality. For example, Figure D.6 shows that when $x = 0$ or $x = 5$ the inequality $2x - 5 < 7$ is satisfied, but when $x = 7$ the inequality $2x - 5 < 7$ is not satisfied.

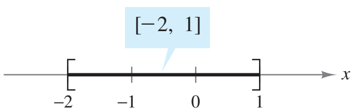
EXAMPLE 3 Solving a Double Inequality

Solve $-3 \le 2 - 5x \le 12$.

Solution

$$
\begin{aligned}
-3 \le \quad 2 - 5x \quad &\le 12 && \text{Write original inequality.}\\
-3 - 2 \le 2 - 5x - 2 &\le 12 - 2 && \text{Subtract 2 from each part.}\\
-5 \le \quad -5x \quad &\le 10 && \text{Simplify.}\\
\frac{-5}{-5} \ge \frac{-5x}{-5} &\ge \frac{10}{-5} && \text{Divide each part by } -5 \text{ and reverse both inequalities.}\\
1 \ge \quad x \quad &\ge -2 && \text{Simplify.}
\end{aligned}
$$

The solution set is $[-2, 1]$, as shown in Figure D.7.



Solution set of $-3 \le 2 - 5x \le 12$
Figure D.7

The inequalities in Examples 2 and 3 are **linear inequalities**—that is, they involve first-degree polynomials. To solve inequalities involving polynomials of higher degree, use the fact that a polynomial can change signs *only* at its real **zeros** (the x-values that make the polynomial equal to zero). Between two consecutive real zeros, a polynomial must be either entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into **test intervals** in which the polynomial has no sign changes. So, if a polynomial has the factored form

$$(x - r_1)(x - r_2) \cdot \cdot \cdot (x - r_n), \qquad r_1 < r_2 < r_3 < \cdot \cdot \cdot < r_n$$

the test intervals are

$$(-\infty, r_1), \quad (r_1, r_2), \ldots, \quad (r_{n-1}, r_n), \quad \text{and} \quad (r_n, \infty).$$

To determine the sign of the polynomial in each test interval, you need to test only *one value* from the interval.

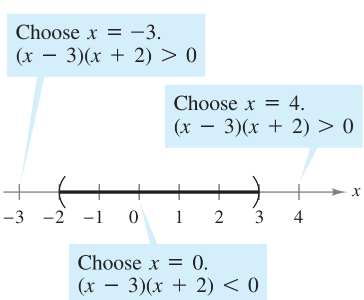
EXAMPLE 4 Solving a Quadratic Inequality

Solve $x^2 < x + 6$.

Solution

$$
\begin{aligned}
x^2 &< x + 6 && \text{Write original inequality.}\\
x^2 - x - 6 &< 0 && \text{Write in general form.}\\
(x - 3)(x + 2) &< 0 && \text{Factor.}
\end{aligned}
$$

The polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros. So, you can solve the inequality by testing the sign of $x^2 - x - 6$ in each of the test intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. To test an interval, choose any number in the interval and compute the sign of $x^2 - x - 6$. After doing this, you will find that the polynomial is positive for all real numbers in the first and third intervals and negative for all real numbers in the second interval. The solution of the original inequality is therefore $(-2, 3)$, as shown in Figure D.8.



Testing an interval
Figure D.8

Absolute Value and Distance

If a is a real number, the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

The absolute value of a number cannot be negative. For example, let $a = -4$. Then, because $-4 < 0$, you have

$$|a| = |-4| = -(-4) = 4.$$

Remember that the symbol $-a$ does not necessarily mean that $-a$ is negative.

Operations with Absolute Value

Let a and b be real numbers and let n be a positive integer.

1. $|ab| = |a|\,|b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

3. $|a| = \sqrt{a^2}$ **4.** $|a^n| = |a|^n$

NOTE You are asked to prove these properties in Exercises 73, 75, 76, and 77.

Properties of Inequalities and Absolute Value

Let a and b be real numbers and let k be a positive real number.

1. $-|a| \leq a \leq |a|$
2. $|a| \leq k$ if and only if $-k \leq a \leq k$.
3. $|a| \geq k$ if and only if $a \leq -k$ or $a \geq k$.
4. *Triangle Inequality:* $|a + b| \leq |a| + |b|$

Properties 2 and 3 are also true if $\leq$ is replaced by $<$.

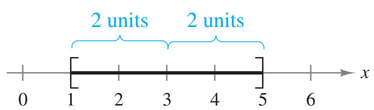
EXAMPLE 5 Solving an Absolute Value Inequality

Solve $|x - 3| \leq 2$.

Solution Using the second property of inequalities and absolute value, you can rewrite the original inequality as a double inequality.

$$-2 \leq x - 3 \leq 2 \qquad \text{Write as double inequality.}$$
$$-2 + 3 \leq x - 3 + 3 \leq 2 + 3 \qquad \text{Add 3 to each part.}$$
$$1 \leq x \leq 5 \qquad \text{Simplify.}$$

The solution set is $[1, 5]$, as shown in Figure D.9.



Solution set of $|x - 3| \leq 2$
Figure D.9

EXAMPLE 6 A Two-Interval Solution Set

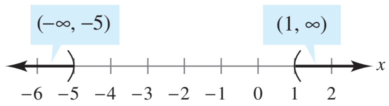
Solve $|x + 2| > 3$.

Solution Using the third property of inequalities and absolute value, you can rewrite the original inequality as two linear inequalities.

$$x + 2 < -3 \quad \text{or} \quad x + 2 > 3$$

$$x < -5 \qquad\qquad x > 1$$

The solution set is the union of the disjoint intervals $(-\infty, -5)$ and $(1, \infty)$, as shown in Figure D.10.



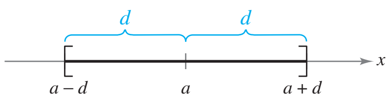
Solution set of $|x + 2| > 3$

Figure D.10

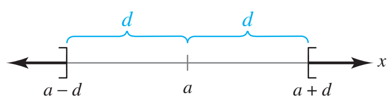
Examples 5 and 6 illustrate the general results shown in Figure D.11. Note that if $d > 0$, the solution set for the inequality $|x - a| \le d$ is a *single* interval, whereas the solution set for the inequality $|x - a| \ge d$ is the union of *two* disjoint intervals.

The **distance between two points** a and b on the real number line is given by

$$d = |a - b| = |b - a|.$$



Solution set of $|x - a| \le d$



Solution set of $|x - a| \ge d$

Figure D.11

The **directed distance from *a* to *b*** is $b - a$ and the **directed distance from *b* to *a*** is $a - b$, as shown in Figure D.12.

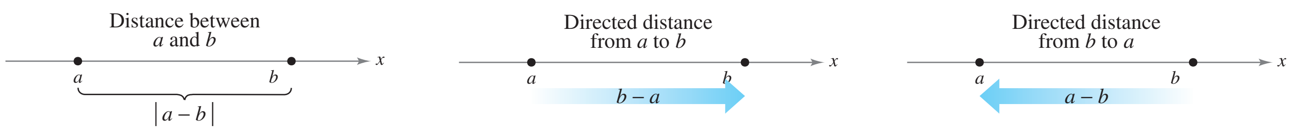


Figure D.12

EXAMPLE 7 Distance on the Real Number Line

a. The distance between -3 and 4 is

$$|4 - (-3)| = |7| = 7 \quad \text{or} \quad |-3 - 4| = |-7| = 7.$$

(See Figure D.13.)

b. The directed distance from -3 to 4 is $4 - (-3) = 7$.

c. The directed distance from 4 to -3 is $-3 - 4 = -7$.

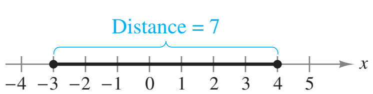


Figure D.13

The **midpoint** of an interval with endpoints a and b is the average value of a and b. That is,

$$\text{Midpoint of interval } (a, b) = \frac{a + b}{2}.$$

To show that this is the midpoint, you need only show that $(a + b)/2$ is equidistant from a and b.

EXERCISES FOR APPENDIX D.1

In Exercises 1–10, determine whether the real number is rational or irrational.

1. 0.7 **2.** -3678

3. $\frac{3\pi}{2}$ **4.** $3\sqrt{2} - 1$

5. $4.3\overline{451}$ **6.** $\frac{22}{7}$

7. $\sqrt[3]{64}$ **8.** $0.\overline{8177}$

9. $4\frac{5}{8}$ **10.** $(\sqrt{2})^3$

In Exercises 11–14, write the repeating decimal as a ratio of two integers using the following procedure. If $x = 0.6363\ldots$, then $100x = 63.6363\ldots$. Subtracting the first equation from the second produces $99x = 63$ or $x = \frac{63}{99} = \frac{7}{11}$.

11. $0.\overline{36}$ **12.** $0.3\overline{18}$

13. $0.\overline{297}$ **14.** $0.\overline{9900}$

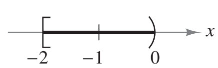
15. Given $a < b$, determine which of the following are true.

(a) $a + 2 < b + 2$ (b) $5b < 5a$

(c) $5 - a > 5 - b$ (d) $\frac{1}{a} < \frac{1}{b}$

(e) $(a - b)(b - a) > 0$ (f) $a^2 < b^2$

16. Complete the table with the appropriate interval notation, set notation, and graph on the real number line.

Interval Notation	Set Notation	Graph
		
$(-\infty, -4]$		
	$\{x\colon 3 \le x \le \frac{11}{2}\}$	
$(-1, 7)$		

In Exercises 17–20, verbally describe the subset of real numbers represented by the inequality. Sketch the subset on the real number line, and state whether the interval is bounded or unbounded.

17. $-3 < x < 3$ **18.** $x \ge 4$

19. $x \le 5$ **20.** $0 \le x < 8$

In Exercises 21–24, use inequality and interval notation to describe the set.

21. y is at least 4.

22. q is nonnegative.

23. The interest rate r on loans is expected to be greater than 3% and no more than 7%.

24. The temperature T is forecast to be above 90°F today.

In Exercises 25–44, solve the inequality and graph the solution on the real number line.

25. $2x - 1 \ge 0$ **26.** $3x + 1 \ge 2x + 2$

27. $-4 < 2x - 3 < 4$ **28.** $0 \le x + 3 < 5$

29. $\frac{x}{2} + \frac{x}{3} > 5$ **30.** $x > \frac{1}{x}$

31. $|x| < 1$ **32.** $\frac{x}{2} - \frac{x}{3} > 5$

33. $\left|\frac{x - 3}{2}\right| \ge 5$ **34.** $\left|\frac{x}{2}\right| > 3$

35. $|x - a| < b,\ b > 0$ **36.** $|x + 2| < 5$

37. $|2x + 1| < 5$ **38.** $|3x + 1| \ge 4$

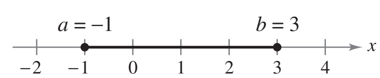
39. $\left|1 - \frac{2}{3}x\right| < 1$ **40.** $|9 - 2x| < 1$

41. $x^2 \le 3 - 2x$ **42.** $x^4 - x \le 0$

43. $x^2 + x - 1 \le 5$ **44.** $2x^2 + 1 < 9x - 3$

In Exercises 45–48, find the directed distance from a to b, the directed distance from b to a, and the distance between a and b.

45.



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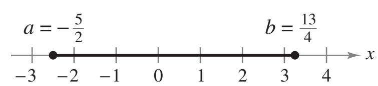


47. (a) $a = 126,\ b = 75$

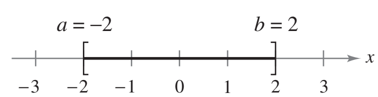
(b) $a = -126,\ b = -75$

48. (a) $a = 9.34,\ b = -5.65$

(b) $a = \frac{16}{5},\ b = \frac{112}{75}$

In Exercises 49–54, use absolute value notation to define the interval or pair of intervals on the real number line.

49.



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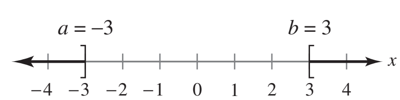

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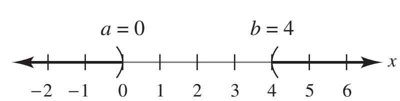

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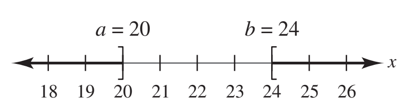


53. (a) All numbers that are at most 10 units from 12.

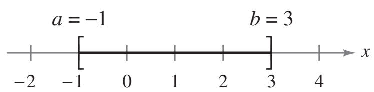
(b) All numbers that are at least 10 units from 12.

54. (a) y is at most two units from a.

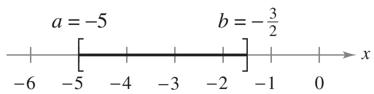
(b) y is less than δ units from c.

In Exercises 55–58, find the midpoint of the interval.

55.



56.



57. (a) $[7, 21]$

(b) $[8.6, 11.4]$

58. (a) $[-6.85, 9.35]$

(b) $[-4.6, -1.3]$

59. ***Profit*** The revenue R from selling x units of a product is

$$R = 115.95x$$

and the cost C of producing x units is

$$C = 95x + 750.$$

To make a (positive) profit, R must be greater than C. For what values of x will the product return a profit?

60. ***Fleet Costs*** A utility company has a fleet of vans. The annual operating cost C (in dollars) of each van is estimated to be

$$C = 0.32m + 2300$$

where m is measured in miles. The company wants the annual operating cost of each van to be less than \$10,000. To do this, m must be less than what value?

61. ***Fair Coin*** To determine whether a coin is fair (has an equal probability of landing tails up or heads up), you toss the coin 100 times and record the number of heads x. The coin is declared unfair if

$$\left|\frac{x - 50}{5}\right| \geq 1.645.$$

For what values of x will the coin be declared unfair?

62. ***Daily Production*** The estimated daily oil production p at a refinery is

$$|p - 2{,}250{,}000| < 125{,}000$$

where p is measured in barrels. Determine the high and low production levels.

In Exercises 63 and 64, determine which of the two real numbers is greater.

63. (a) π or $\frac{355}{113}$

(b) π or $\frac{22}{7}$

64. (a) $\frac{224}{151}$ or $\frac{144}{97}$

(b) $\frac{73}{81}$ or $\frac{6427}{7132}$

65. ***Approximation—Powers of 10*** Light travels at the speed of 2.998×10^8 meters per second. Which best estimates the distance in meters that light travels in a year?

(a) 9.5×10^5 (b) 9.5×10^{15}

(c) 9.5×10^{12} (d) 9.6×10^{16}

66. ***Writing*** The accuracy of an approximation to a number is related to how many significant digits there are in the approximation. Write a definition for significant digits and illustrate the concept with examples.

True or False? **In Exercises 67–72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. The reciprocal of a nonzero integer is an integer.

68. The reciprocal of a nonzero rational number is a rational number.

69. Each real number is either rational or irrational.

70. The absolute value of each real number is positive.

71. If $x < 0$, then $\sqrt{x^2} = -x$.

72. If a and b are any two distinct real numbers, then $a < b$ or $a > b$.

In Exercises 73–80, prove the property.

73. $|ab| = |a||b|$

74. $|a - b| = |b - a|$

$[Hint\text{: } (a - b) = (-1)(b - a)]$

75. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0$

76. $|a| = \sqrt{a^2}$

77. $|a^n| = |a|^n, \quad n = 1, 2, 3, \ldots$

78. $-|a| \leq a \leq |a|$

79. $|a| \leq k$, if and only if $-k \leq a \leq k, \quad k > 0.$

80. $|a| \geq k$ if and only if $a \leq -k$ or $a \geq k, \quad k > 0.$

81. Find an example for which $|a - b| > |a| - |b|$, and an example for which $|a - b| = |a| - |b|$. Then prove that $|a - b| \geq |a| - |b|$ for all a, b.

82. Show that the maximum of two numbers a and b is given by the formula

$$\max(a, b) = \tfrac{1}{2}(a + b + |a - b|).$$

Derive a similar formula for $\min(a, b)$.